

The Weihrauch degree of Ramsey's Theorem for two colors

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Purpose of the study

Use Weihrauch degrees to classify mathematical theorems according to their computational content.

Idea

Regard a theorem as a map:

Example

- ▶ A Π_2 theorem: “ $(\forall x \in X)(\exists y \in Y)(x, y) \in A$ ” can be seen as a multivalued map $f : x \mapsto \{y : (x, y) \in A\}$.

Contents

- ▶ Introduction to Weihrauch Degrees
- ▶ Variants of Ramsey's Theorem
- ▶ Idempotency and Parallelization

Contents

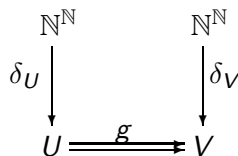
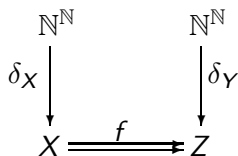
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Represented Sets and Realizers

$$X \xRightarrow{f} Z$$

$$U \xRightarrow{g} V$$

Represented Sets and Realizers

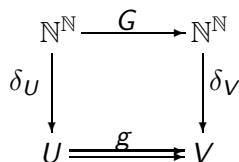
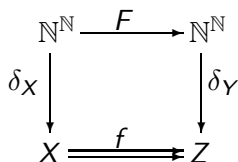


Represented Sets and Realizers

$$\begin{array}{ccc}
 \mathbb{N}^{\mathbb{N}} & \xrightarrow{F} & \mathbb{N}^{\mathbb{N}} \\
 \delta_X \downarrow & & \downarrow \delta_Y \\
 X & \xRightarrow{f} & Z
 \end{array}$$

$$\begin{array}{ccc}
 \mathbb{N}^{\mathbb{N}} & \xrightarrow{G} & \mathbb{N}^{\mathbb{N}} \\
 \delta_U \downarrow & & \downarrow \delta_V \\
 U & \xRightarrow{g} & V
 \end{array}$$

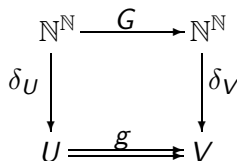
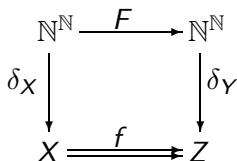
Represented Sets and Realizers



- ▶ (X, δ_X) is a **represented set** if $\delta_X : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow X$ is surjective
- ▶ F is a **realizer** of f if for all $p \in \text{dom}(f\delta_X)$ we get $\delta_Y F(p) \in f\delta_X(p)$ (noted by $F \vdash f$)

If $\delta(p) = x$ then we say p is a name of the object x .

Weihrauch Degree



- ▶ f is **strongly Weihrauch reducible** to g if there exist two computable functions H and K such that $H \circ G \circ K \vdash f$ for all $G \vdash g$ (noted be $f \leq_{sW} g$)
- ▶ f is (weakly) **Weihrauch reducible** to g if there exist two computable functions H and K such that $H \langle \text{id}, G \circ K \rangle \vdash f$ for all $G \vdash g$ (noted be $f \leq_W g$)

Invariance Under Representations

Definition

If we have two representations δ_1 and δ_2 of a set X then δ_1 is said reducible to δ_2 , noted by $\delta_1 \leq \delta_2$, if there is a computable function $\Phi : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$ such that $\delta_1(p) = \delta_2(\Phi(p))$ for all $p \in \text{dom}(\delta_1)$

Lemma

Weihrauch degrees are invariant under equivalent representations.

Tupling Functions and the Limit Map

Definition

Let $(p_i)_{i \in \mathbb{N}}$ be a sequence in Baire space. We define the following:

- ▶ $\langle p_i, p_j \rangle(2n) = p_i(n)$ and $\langle p_i, p_j \rangle(2n + 1) = p_j(n)$
- ▶ $\langle p_0, p_1, \dots, p_n \rangle = \langle p_0, \langle p_1, \dots, p_n \rangle \rangle$
- ▶ $\langle p_0, p_1, \dots \rangle \langle n, k \rangle = p_n(k)$
- ▶ $\lim : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}; \lim \langle p_0, p_1, \dots \rangle(n) = \lim_{i \rightarrow \infty} p_i(n)$

Operators

Let $f : \subseteq (X, \delta_X) \rightrightarrows (Y, \delta_Y)$ be a multivalued function. Then we define

- ▶ the **parallelization** $\hat{f} : \subseteq (X^{\mathbb{N}}, \delta_X^{\mathbb{N}}) \rightrightarrows (Y^{\mathbb{N}}, \delta_Y^{\mathbb{N}})$ of f by

$$\hat{f}(x_i)_{i \in \mathbb{N}} := \times_{i=0}^{\infty} f(x_i)$$

for all $(x_i) \in X^{\mathbb{N}}$, where $\delta^{\mathbb{N}} : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow X^{\mathbb{N}}$ is defined by

$$\delta^{\mathbb{N}} \langle p_0, p_1, \dots \rangle := (\delta(p_i))_{i \in \mathbb{N}}$$

- ▶ the **jump** $f' : \subseteq (X, \delta'_X) \rightrightarrows (Y, \delta_Y)$ of f by $f'(x) = f(x)$ and $\delta' := \delta \circ \lim$
- ▶ for $n \geq 1$; $f^n : \subseteq (X^n, \delta^n) \rightrightarrows (Y^n, \delta^n)$ where $\delta^n \langle p_0, \dots, p_n \rangle = (\delta(p_0), \dots, \delta(p_n))$

Facts

Let f and g be multivalued functions on represented spaces. Then

- ▶ $f \leq_W \widehat{f}$
- ▶ $f \leq_W g \implies \widehat{f} \leq_W \widehat{g}$
- ▶ $\widehat{f} \equiv_W \widehat{\widehat{f}}$
- ▶ $f \leq_{sW} f'$
- ▶ $f \leq_{sW} g \implies f' \leq_{sW} g'$

Invariance Principles

Lemma

Let f and g be multivalued functions on represented spaces such that $f \leq_W g$. Let $n \in \mathbb{N}$.

- ▶ (Computable Invariance Principle) *If g has a realizer that maps computable inputs to computable outputs, then f has a realizer that maps computable inputs to computable outputs.*

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Ramsey Theory

Definition

Given $l \geq 1$ and $k \geq 2$ we define

- ▶ $[\mathbb{N}]^l := \{\text{size } l \text{ subsets of } \mathbb{N}\}$
 - $[\mathbb{N}]^1 = \{\{0\}, \{1\}, \{2\}, \{3\}, \dots\}$
 - $[\mathbb{N}]^2 = \{\{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 3\}, \{1, 3\}, \{2, 3\}, \{0, 4\}, \dots\}$
- ▶ a coloring $c : [\mathbb{N}]^l \rightarrow \{0, 1, 2, \dots, k - 1\}$

Theorem (Ramsey's Theorem)

Given $l, k \geq 1$ and a coloring c , there is an infinite subset M of \mathbb{N} on which c is constant on $[M]^l$

Such sets M will be called homogeneous and we write $c(M) = x$ if x is the constant value of c on M .

Ramsey's Theorem as a Map

Definition

We define the following:

- ▶ $\mathcal{C}_{l,k}$ denotes the set of all $c : [\mathbb{N}]^l \rightarrow \{0, 1, 2, \dots, k-1\}$
- ▶ $\text{RT}_{l,k} : \mathcal{C}_{l,k} \rightrightarrows 2^{\mathbb{N}}; c \mapsto \{M : M \text{ is homogeneous for } c\}$

Sets are represented by their characteristic function and $\mathcal{C}_{l,k}$ can be represented in the following way: $\delta_{\mathcal{C}_{l,k}}(p) = c$ if for all $\{i_1, \dots, i_l\} \in [\mathbb{N}]^l$ we have $c\{i_1, \dots, i_l\} = x$ iff $p\langle i_1, \dots, i_l \rangle = x$

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$\{i_1, \dots, i_l\} \in [\mathbb{N}]^l$ we have $c\{i_1, \dots, i_l\} = x$ iff $p\langle i_1, \dots, i_l \rangle = x$

The following maps are also very interesting

- ▶ $\text{MRT}_{l,k} : \mathcal{C}_{l,k} \rightrightarrows 2^{\mathbb{N}};$
 $c \mapsto \{M : M \text{ is a maximal homogeneous set for } c\}$
- ▶ $\text{CRT}_{l,k} : \mathcal{C}_{l,k} \rightrightarrows \mathbb{N} \times 2^{\mathbb{N}};$
 $c \mapsto \{(x, M) : M \text{ is an homogeneous set with } c(M) = x\}$

Finite Intersection

Lemma

Given $n \in \mathbb{N}$ and c_1, \dots, c_n in $\mathcal{C}_{l,k}$, we get $\bigcap_{i=1}^n \text{RT}_{l,k}(c_i) \neq \emptyset$.

Proof idea.

We construct a map $t : (\mathcal{C}_{l,k})^n \rightarrow \mathcal{C}_{l,k^n}; (c_1, \dots, c_n) \mapsto c$ such that $\text{RT}_{l,k^n}(c) = \bigcap_{i=1}^n \text{RT}_{l,k}(c_i)$. And we apply Ramsey's Theorem itself. □

Definition

$$\begin{aligned} \bigcap^n \text{RT}_{l,k} : (\mathcal{C}_{l,k})^n &\rightrightarrows 2^{\mathbb{N}}; \\ (c_1, \dots, c_n) &\mapsto \{M : M \text{ is homogeneous for each } c_i\} \end{aligned}$$

Bolzano-Weierstrass and Ramsey Theorems

Definition

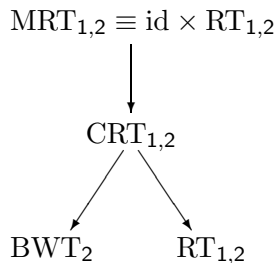
We define the Bolzano-Weierstrass map for $\{0, 1\}$ as the following:

$$\text{BWT}_2 : \{0, 1\}^{\mathbb{N}} \rightrightarrows \{0, 1\}; \quad p \mapsto \{x : (\exists^\infty n) \, p(n) = x\}$$

Lemma

- ▶ $\text{BWT}_2 \equiv_W \text{RT}_{1,2} \equiv_W \text{CRT}_{1,2} \equiv_W \text{MRT}_{1,2}$
- ▶ $\text{BWT}_2|_{sW} \text{RT}_{1,2}$
- ▶ $\text{BWT}_2 <_{sW} \text{CRT}_{1,2}$ *and* $\text{RT}_{1,2} <_{sW} \text{CRT}_{1,2}$
- ▶ $\text{CRT}_{1,2} <_{sW} \text{MRT}_{1,2}$
- ▶ $\text{MRT}_{1,2} \equiv_{sW} \text{id} \times \text{RT}_{1,2}$

Strong Reducibility



Jumps and Strong Reducibility

Theorem

$$\text{BWT}'_2 \not\leq_{sW} \text{RT}'_{1,2}$$

Proof.

BWT'_2 maps computable inputs to computable outputs. However there is a Δ_2^0 set which is bi-immune. Hence $\text{RT}'_{1,2}$ maps some computable inputs only to non-computable outputs. By the Computable Invariance Principle $\text{RT}'_{1,2} \not\leq_{sW} \text{BWT}'_2$. We get a strong result for the other direction. □

Omniscience Principle and Ramsey Theorems

$$\text{LLPO} : \subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}; \text{LLPO}(p) \ni \begin{cases} 0 & \text{if } (\forall n \in \mathbb{N}) p(2n) = 0, \\ 1 & \text{if } (\forall n \in \mathbb{N}) p(2n+1) = 0 \end{cases}$$

where $\text{dom}(\text{LLPO}) = \{p \in \mathbb{N}^{\mathbb{N}} : p(k) \neq 0 \text{ for at most one } k\}$

Theorem

$$\text{LLPO} \not\leq_{sW} \text{RT}'_{1,2}$$

Proof idea.

Assuming the contrary will violate the Finite Intersection Lemma. □

The Stable Ramsey Theorem

Definition

Let c be in $\mathcal{C}_{2,2}$, we say that c is **stable** if for all $m \in \mathbb{N}$ the limit $\lim_{n \rightarrow \infty} (c\{n, m\})$ exists. And we define

- ▶ $\text{SRT}_{2,2} : \subseteq \mathcal{C}_{2,2} \Rightarrow 2^{\mathbb{N}}$, where $\text{dom}(\text{SRT}_{2,2}) = \{c : c \text{ is stable}\}$ and $\text{SRT}_{2,2}(c) = \text{RT}_{2,2}(c)$ for all $c \in \text{dom}(\text{SRT}_{2,2})$

Theorem

$$\text{CRT}'_{1,2} \equiv_W \text{SRT}_{2,2}$$

Coin Avoidance and The Limit Map

Theorem (Seetapun and Slaman 1995)

For any computable coloring $c \in \mathcal{C}_{2,2}$ and non-computable set A there is an homogeneous set $M \in \text{RT}_{2,2}(c)$ such that $A \not\leq_T M$.

Theorem

- ▶ $\lim \not\leq_W \text{RT}_{2,2}$
- ▶ $\text{RT}_{2,2} \not\leq_W \text{MRT}'_{1,2}$
- ▶ $\lim <_{sW} \text{MRT}'_{1,2}$
- ▶ $\lim|_W \text{CRT}'_{1,2}$

More Theorems

Definition

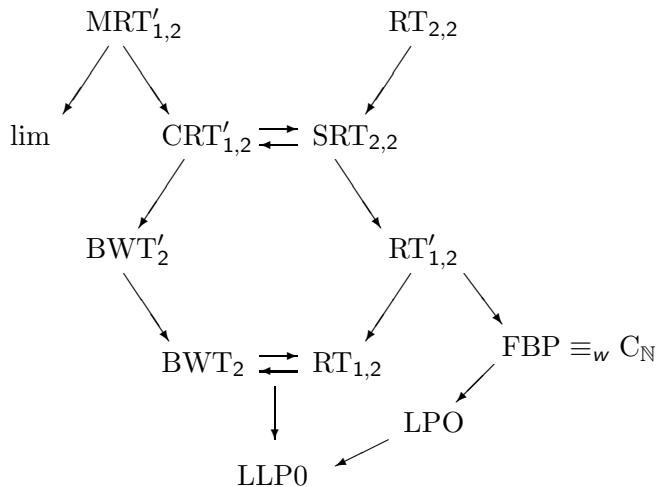
We define the two following maps which are the Finite Boundedness Principle and the Choice on Natural Numbers.

- ▶ $\text{FBP} : \subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}; p \mapsto \{b : (\forall n \in \mathbb{N}) p(n) \leq b\}$
- ▶ $\text{C}_{\mathbb{N}} : \subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}; p \mapsto \{b : (\forall n \in \mathbb{N}) p(n) \neq b\}$

Lemma

- ▶ $\text{FBP} \equiv_w \text{C}_{\mathbb{N}}$
- ▶ $\text{C}_{\mathbb{N}} \leq_w \text{RT}'_{1,2}$

Conclusion



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Idempotency and Parallelization

Definition

Let f be a function on represented spaces. We say that f is:

- ▶ **idempotent** if $f^2 \equiv_W f$
- ▶ **parallelizable** if $\hat{f} \equiv_W f$

Finite Tolerance (Dorais et. al. 2012)

Definition

Let $f : \subseteq (X, \delta_X) \rightrightarrows (Y, \delta_Y)$. We say that f is **finitely tolerant** if there exists a computable function $T : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$ such that for any realizer $F \vdash f$ and any p and q in $\text{dom}(f\delta_X)$, for all $k \in \mathbb{N}$

- (1) $(\forall n) p(n+k) = q(n)$ implies
- (2) $r = F(p) \implies \delta_Y T\langle r, k \rangle \in f\delta_X(q)$

Definition

A function $f : \subseteq (X, \delta_X) \rightrightarrows (Y, \delta_Y)$ is **totally represented** if δ_X is total.

Squashing Theorem (Dorais et. al. 2012)

Example

$RT_{l,k}$ and BWT_n are finitely tolerant and totally represented.

Theorem

If f is finitely tolerant, totally represented and idempotent then f is parallelizable.

Parallelization (Dorais et. al. 2012)

Theorem

$\text{RT}_{l,k}$ is not parallelizable.

$$\blacktriangleright \widehat{\text{RT}_{l,2}} \not\leq_W \text{RT}_{l,k}$$

Corollary

$\text{RT}_{l,k}$ is not idempotent.

Separation for Different Size

Theorem

$(RT_{l,k})^n <_{sW} RT_{l+1,2}$ and $(RT_{l,k})^n <_W RT_{l+1,2}$

- ▶ $RT_{l,k} <_W RT_{l+1,k}$
- ▶ $RT_{3,2} <_{sW} RT_{4,2}$ (Dorais et. al. 2012)

Question

$\widehat{RT_{l,k}} \not<_W RT_{l+1,k}?$

Separation for Different Color

Theorem (Dorais et. al. 2012)

$$\text{RT}_{l,k} <_{sW} \text{RT}_{l,k+1} \text{ and } \text{RT}_{l,k} <_W \text{RT}_{l,k+1}$$

Question

$$(\text{RT}_{l,k})^n \not\leq_W \text{RT}_{l,k+1}?$$

Theorem

$$(\text{RT}_{l,k})^n \leq_{sW} \cap^n \text{RT}_{l,k} \equiv \text{RT}_{l,k^n}$$



Vasco Brattka, Matthew de Brecht, and Arno Pauly.

Closed choice and a uniform low basis theorem.

Annals of Pure and Applied Logic, 163:986–1008, 2012.



Vasco Brattka and Guido Gherardi.

Weihrauch degrees, omniscience principles and weak computability.

The Journal of Symbolic Logic, 76(1):143–176, 2011.



Vasco Brattka, Guido Gherardi, and Alberto Marcone.

The Bolzano-Weierstrass theorem is the jump of weak König's lemma.

Annals of Pure and Applied Logic, 163:623–655, 2012.



François G. Dorais, Damir D. Dzhalalov, Jeffry L. Hirst, Joseph R. Mileti, and Paul Shafer.

On the uniform relationships between combinatorial problems.

2012.

THANK YOU