The Weihrauch degree of Ramsey's Theorem for two colors

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Puspose of the study

Use Weihrauch degrees to classify mathematical theorems according to their computational content.

Idea

Regard a theorem as a map:

Example

- A Π_2 theorem: "$(\forall x \in X)(\exists y \in Y)(x, y) \in A$" can be seen as a multivalued map $f : x \mapsto \{y : (x, y) \in A\}$.

Contents

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Represented Sets and Realizers

$$X \overset{f}{\rightrightarrows} Z \qquad U \overset{g}{\rightrightarrows} V$$

Represented Sets and Realizers

$$
\begin{tikzcd}
\mathbb{N}^{\mathbb{N}} \arrow[d, "\delta_X"'] & \mathbb{N}^{\mathbb{N}} \arrow[d, "\delta_Y"] \\
X \arrow[r, Rightarrow, "f"] & Z
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{N}^{\mathbb{N}} \arrow[d, "\delta_U"'] & \mathbb{N}^{\mathbb{N}} \arrow[d, "\delta_V"] \\
U \arrow[r, Rightarrow, "g"] & V
\end{tikzcd}
$$

Represented Sets and Realizers

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Represented Sets and Realizers

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- (X, δ_X) is a represented set if $\delta_X :\subseteq \mathbb{N}^{\mathbb{N}} \to X$ is surjective
- F is a realizer of f if for all $p \in \mathrm{dom}(f\delta_X)$ we get $\delta_Y F(p) \in f\delta_X(p)$ (noted by $F \vdash f$)

If $\delta(p) = x$ then we say p is a name of the object x.

Weihrauch Degree

$$\begin{tikzcd} \mathbb{N}^{\mathbb{N}} \arrow[r, "F"] \arrow[d, "\delta_X"'] & \mathbb{N}^{\mathbb{N}} \arrow[d, "\delta_Y"] \\ X \arrow[r, Rightarrow, "f"] & Z \end{tikzcd} \qquad \begin{tikzcd} \mathbb{N}^{\mathbb{N}} \arrow[r, "G"] \arrow[d, "\delta_U"'] & \mathbb{N}^{\mathbb{N}} \arrow[d, "\delta_V"] \\ U \arrow[r, Rightarrow, "g"] & V \end{tikzcd}$$

- f is strongly Weihrauch reducible to g if there exist two computable functions H and K such that $H \circ G \circ K \vdash f$ for all $G \vdash g$ (noted be $f \leq_{sW} g$)
- f is (weakly) Weihrauch reducible to g if there exist two computable functions H and K such that $H\langle \mathrm{id}, G \circ K\rangle \vdash f$ for all $G \vdash g$ (noted be $f \leq_W g$)

Invariance Under Representations

Definition

If we have two representations δ_1 and δ_2 of a set X then δ_1 is said reducible to δ_2, noted by $\delta_1 \leq \delta_2$, if there is a computable function $\Phi :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that $\delta_1(p) = \delta_2\Phi(p)$ for all $p \in \mathrm{dom}(\delta_1)$

Lemma

Weihrauch degrees are invariant under equivalent representations.

Tupling Functions and the Limit Map

Definition

Let $(p_i)_{i\in\mathbb{N}}$ be a sequence in Baire space. We define the following:

- $\langle p_i, p_j\rangle(2n) = p_i(n)$ and $\langle p_i, p_j\rangle(2n+1) = p_j(n)$
- $\langle p_0, p_1, ..., p_n\rangle = \langle p_0, \langle p_1, ..., p_n\rangle\rangle$
- $\langle p_0, p_1, ...\rangle\langle n, k\rangle = p_n(k)$
- $\lim :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}; \lim\langle p_0, p_1, ...\rangle(n) = \lim_{i\to\infty} p_i(n)$

Operators

Let $f :\subseteq (X, \delta_X) \rightrightarrows (Y, \delta_Y)$ be a multivalued function. Then we define

- the parallelization $\widehat{f} :\subseteq (X^{\mathbb{N}}, \delta_X^{\mathbb{N}}) \rightrightarrows (Y^{\mathbb{N}}, \delta_Y^{\mathbb{N}})$ of f by

  $$\widehat{f}(x_i)_{i \in \mathbb{N}} := \times_{i=0}^{\infty} f(x_i)$$

  for all $(x_i) \in X^{\mathbb{N}}$, where $\delta^{\mathbb{N}} :\subseteq \mathbb{N}^{\mathbb{N}} \to X^{\mathbb{N}}$ is defined by $\delta^{\mathbb{N}}\langle p_0, p_1, ...\rangle := (\delta(p_i))_{i \in \mathbb{N}}$
- the jump $f' :\subseteq (X, \delta'_X) \rightrightarrows (Y, \delta_Y)$ of f by $f'(x) = f(x)$ and $\delta' := \delta \circ \lim$
- for $n \geq 1$; $f^n :\subseteq (X^n, \delta^n) \rightrightarrows (Y^n, \delta^n)$ where $\delta^n\langle p_0, ..., p_n\rangle = (\delta(p_0), ..., \delta(p_n))$

Facts

Let f and g be multivalued functions on represented spaces. Then

- $f \leq_W \widehat{f}$
- $f \leq_W g \Longrightarrow \widehat{f} \leq_W \widehat{g}$
- $\widehat{f} \equiv_W \widehat{\widehat{f}}$
- $f \leq_{sW} f'$
- $f \leq_{sW} g \Longrightarrow f' \leq_{sW} g'$

Invariance Principles

Lemma

Let f and g be multivalued functions on represented spaces such that $f \leq_W g$. Let $n \in \mathbb{N}$.

- (Computable Invariance Principle) *If g has a realizer that maps computable inputs to computable outputs, then f has a realizer that maps computable inputs to computable outputs.*

Contents

Ramsey Theory

Definition

Given $l \geq 1$ and $k \geq 2$ we define

- $[\mathbb{N}]^l := \{\text{size } l \text{ subsets } of\ \mathbb{N}\}$
  - $[\mathbb{N}]^1 = \{\{0\}, \{1\}, \{2\}, \{3\}, ...\}$
  - $[\mathbb{N}]^2 = \{\{0,1\}, \{0,2\}, \{1,2\}, \{0,3\}, \{1,3\}, \{2,3\}, \{0,4\}, ...\}$
- a coloring $c : [\mathbb{N}]^l \to \{0, 1, 2, ..., k-1\}$

Theorem (Ramsey's Theorem)

Given $l, k \geq 1$ and a coloring c, there is an infinite subset M of $\mathbb{N}$ on which c is constant on $[M]^l$

Such sets M will be called homogeneous and we write $c(M) = x$ if x is the constant value of c on M.

Ramsey's Theorem as a Map

Definition

We define the following:

- $\mathcal{C}_{l,k}$ denotes the set of all $c : [\mathbb{N}]^l \to \{0, 1, 2, ..., k-1\}$
- $\mathrm{RT}_{l,k} : \mathcal{C}_{l,k} \rightrightarrows 2^{\mathbb{N}};\ c \mapsto \{M : M \text{ is homogeneous for } c\}$

Sets are represented by their characteristic function and $\mathcal{C}_{l,k}$ can be represented in the following way: $\delta_{\mathcal{C}_{l,k}}(p) = c$ if for all $\{i_1, ..., i_l\} \in [\mathbb{N}]^l$ we have $c\{i_1, ..., i_l\} = x$ iff $p\langle i_1, ..., i_l\rangle = x$

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The following maps are also very interesting

- $\mathrm{MRT}_{l,k} : \mathcal{C}_{l,k} \rightrightarrows 2^{\mathbb{N}}$;
  $c \mapsto \{M : M \text{ is a maximal homogeneous set for } c\}$
- $\mathrm{CRT}_{l,k} : \mathcal{C}_{l,k} \rightrightarrows \mathbb{N} \times 2^{\mathbb{N}}$;
  $c \mapsto \{(x, M) : M \text{ is an homogeneous set with } c(M) = x\}$

Finite Intersection

Lemma

Given $n \in \mathbb{N}$ and $c_1, ..., c_n$ in $\mathcal{C}_{l,k}$, we get $\cap_{i=1}^{n} \mathrm{RT}_{l,k}(c_i) \neq \emptyset$.

Proof idea.

We construct a map $t : (\mathcal{C}_{l,k})^n \to \mathcal{C}_{l,k^n}; (c_1, ..., c_n) \mapsto c$ such that $\mathrm{RT}_{l,k^n}(c) = \cap_{i=1}^{n} \mathrm{RT}_{l,k}(c_i)$. And we apply Ramsey's Theorem itself. $\square$

Definition

$\cap^n \mathrm{RT}_{l,k} : (\mathcal{C}_{l,k})^n \rightrightarrows 2^{\mathbb{N}};$
$(c_1, ..., c_n) \mapsto \{M : M \text{ is homogeneous for each } c_i\}$

Bolzano-Weierstrass and Ramsey Theorems

Definition

We define the Bolzano-Weierstrass map for $\{0,1\}$ as the following:
$\mathrm{BWT}_2 : \{0,1\}^{\mathbb{N}} \rightrightarrows \{0,1\};\ p \mapsto \{x : (\exists^{\infty} n)\ p(n) = x\}$

Lemma

- $\mathrm{BWT}_2 \equiv_W \mathrm{RT}_{1,2} \equiv_W \mathrm{CRT}_{1,2} \equiv_W \mathrm{MRT}_{1,2}$
- $\mathrm{BWT}_2 |_{sW} \mathrm{RT}_{1,2}$
- $\mathrm{BWT}_2 <_{sW} \mathrm{CRT}_{1,2}$ *and* $\mathrm{RT}_{1,2} <_{sW} \mathrm{CRT}_{1,2}$
- $\mathrm{CRT}_{1,2} <_{sW} \mathrm{MRT}_{1,2}$
- $\mathrm{MRT}_{1,2} \equiv_{sW} \mathrm{id} \times \mathrm{RT}_{1,2}$

Strong Reducibility

$$\begin{tikzcd}
 & \mathrm{MRT}_{1,2} \equiv \mathrm{id} \times \mathrm{RT}_{1,2} \arrow[d] & \\
 & \mathrm{CRT}_{1,2} \arrow[dl] \arrow[dr] & \\
\mathrm{BWT}_2 & & \mathrm{RT}_{1,2}
\end{tikzcd}$$

Jumps and Strong Reducibility

Theorem

$\mathrm{BWT}_2' |_{sW} \mathrm{RT}_{1,2}'$

Proof.

BWT_2' maps computable inputs to computable outputs. However there is a Δ_2^0 set which is bi-immune. Hence $\mathrm{RT}_{1,2}'$ maps some computable inputs only to non-computable outputs. By the Computable Invariance Principle $\mathrm{RT}_{1,2}' \not\leq_{sW} \mathrm{BWT}_2'$. We get a strong result for the other direction. □

Omniscience Principle and Ramsey Theorems

$$\mathrm{LLPO} :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}; \mathrm{LLPO}(p) \ni \begin{cases} 0 & \text{if } (\forall n \in \mathbb{N}) p(2n) = 0, \\ 1 & \text{if } (\forall n \in \mathbb{N}) p(2n+1) = 0 \end{cases}$$

where $\mathrm{dom}(\mathrm{LLPO}) = \{p \in \mathbb{N}^{\mathbb{N}} : p(k) \neq 0 \text{ for at most one } k\}$

Theorem

$\mathrm{LLPO} \not\leq_{sW} \mathrm{RT}'_{1,2}$

Proof idea.

Assuming the contrary will violate the Finite Intersection Lemma. $\square$

The Stable Ramsey Theorem

Definition

Let c be in $\mathcal{C}_{2,2}$, we say that c is stable is for all $m \in \mathbb{N}$ the limit $\lim_{n\to\infty}(c\{n,m\})$ exists. And we define

- $\mathrm{SRT}_{2,2} :\subseteq \mathcal{C}_{2,2} \rightrightarrows 2^{\mathbb{N}}$, where $\mathrm{dom}(\mathrm{SRT}_{2,2}) = \{c : c \text{ is stable}\}$ and $\mathrm{SRT}_{2,2}(c) = \mathrm{RT}_{2,2}(c)$ for all $c \in \mathrm{dom}(\mathrm{SRT}_{2,2})$

Theorem

$\mathrm{CRT}'_{1,2} \equiv_W \mathrm{SRT}_{2,2}$

Coin Avoidance and The Limit Map

Theorem (Seetapun and Slaman 1995)

For any computable coloring $c \in \mathcal{C}_{2,2}$ *and non-computable set* A *there is an homogeneous set* $M \in \mathrm{RT}_{2,2}(c)$ *such that* $A \not\leq_T M$.

Theorem

- $\lim \not\leq_W \mathrm{RT}_{2,2}$
- $\mathrm{RT}_{2,2} \not\leq_W \mathrm{MRT}'_{1,2}$
- $\lim <_{sW} \mathrm{MRT}'_{1,2}$
- $\lim |_W \mathrm{CRT}'_{1,2}$

More Theorems

Definition

We define the two following maps which are the Finite Boundedness Principle and the Choice on Natural Numbers.

- $\mathrm{FBP} :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}; p \mapsto \{b : (\forall n \in \mathbb{N}) p(n) \leq b\}$
- $\mathrm{C}_{\mathbb{N}} :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}; p \mapsto \{b : (\forall n \in \mathbb{N}) p(n) \neq b)\}$

Lemma

- $\mathrm{FBP} \equiv_{w} \mathrm{C}_{\mathbb{N}}$
- $\mathrm{C}_{\mathbb{N}} \leq_{W} \mathrm{RT}'_{1,2}$

Conclusion

$$\begin{tikzcd}
 & \mathrm{MRT}'_{1,2} \arrow[dl] \arrow[dr] & & & & \mathrm{RT}_{2,2} \arrow[dl] & \\
\lim & & \mathrm{CRT}'_{1,2} \arrow[dl] & \rightleftarrows & \mathrm{SRT}_{2,2} \arrow[dr] & & \\
 & \mathrm{BWT}'_2 \arrow[dr] & & & & \mathrm{RT}'_{1,2} \arrow[dl] \arrow[dr] & \\
 & & \mathrm{BWT}_2 & \rightleftarrows \arrow[dd] & \mathrm{RT}_{1,2} & & \mathrm{FBP} \equiv_W \mathrm{C}_{\mathbb{N}} \arrow[dl] \\
 & & & & & \mathrm{LPO} \arrow[dll] & \\
 & & & \mathrm{LLP0} & & &
\end{tikzcd}$$

Contents

Idempotency and Parallelization

Definition

Let f be a function on represented spaces. We say that f is:

- idempotent if $f^2 \equiv_W f$
- parallelizable if $\widehat{f} \equiv_W f$

Finite Tolerance (Dorais et. al. 2012)

Definition

Let $f :\subseteq (X, \delta_X) \rightrightarrows (Y, \delta_Y)$. We say that f is finitely tolerant if there exists a computable function $T :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that for any realizer $F \vdash f$ and any p and q in $\mathrm{dom}(f\delta_X)$, for all $k \in \mathbb{N}$

(1) $(\forall n)\ p(n+k) = q(n)$ implies

(2) $r = F(p) \Longrightarrow \delta_Y T\langle r, k\rangle \in f\delta_X(q)$

Definition

A function $f :\subseteq (X, \delta_X) \rightrightarrows (Y, \delta_Y)$ is totally represented if δ_X is total.

Squashing Theorem (Dorais et. al. 2012)

Example

$\mathrm{RT}_{l,k}$ and $\mathrm{BWT_n}$ are finitely tolerant and totally represented.

Theorem

If f is finitely tolerant, totally represented and idempotent then f is parallelizable.

Parallelization (Dorais et. al. 2012)

Theorem

$\mathrm{RT}_{l,k}$ *is not parallelizable.*

- $\widehat{\mathrm{RT}_{l,2}} \nleq_W \mathrm{RT}_{l,k}$

Corollary

$\mathrm{RT}_{l,k}$ *is not idempotent.*

Separation for Different Size

Theorem

$(\mathrm{RT}_{l,k})^n <_{sW} \mathrm{RT}_{l+1,2}$ *and* $(\mathrm{RT}_{l,k})^n <_{W} \mathrm{RT}_{l+1,2}$

- $\mathrm{RT}_{l,k} <_W \mathrm{RT}_{l+1,k}$
- $\mathrm{RT}_{3,2} <_{sW} \mathrm{RT}_{4,2}$ (Dorais et. al. 2012)

Question

$\widehat{\mathrm{RT}_{l,k}} \not\leq_W \mathrm{RT}_{l+1,k}$?

Separation for Different Color

Theorem (Dorais et. al. 2012)

$\mathrm{RT}_{l,k} <_{sW} \mathrm{RT}_{l,k+1}$ *and* $\mathrm{RT}_{l,k} <_{W} \mathrm{RT}_{l,k+1}$

Question

$(\mathrm{RT}_{l,k})^n \not\leq_W \mathrm{RT}_{l,k+1}$?

Theorem

$(\mathrm{RT}_{l,k})^n \leq_{sW} \cap^n \mathrm{RT}_{l,k} \equiv \mathrm{RT}_{l,k^n}$

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THANK YOU